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From AKNS to derivative NLS hierarchies via deformations of associative products

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Abstract

Using deformations of associative products, derivative nonlinear Schrödinger (DNLS) hierarchies are recovered as AKNS-type hierarchies. Since the latter can also be formulated as Gelfand–Dickey-type Lax hierarchies, a recently developed method to obtain ‘functional representations’ can be applied. We actually consider hierarchies with dependent variables in any (possibly noncommutative) associative algebra, e.g., an algebra of matrices of functions. This also covers the case of hierarchies of coupled derivative NLS equations.

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1. Introduction

An AKNS hierarchy (see [1–7], for example) is given by (a multi-component generalization of)

$$\partial_{t_n}(V) = [(\zeta^n V)_{\geq k}, V] \quad n = 1, 2, \dots \quad (1.1)$$

with independent variables t_n , an indeterminate ζ , a formal power series V in ζ^{-1} with coefficients in a Lie algebra \mathcal{G} and a projection $(\cdot)_{\geq k}$ to terms containing only powers $\geq k$ of ζ , where $k = 0$ or $k = 1$. The Lie bracket used in (1.1) may depend nontrivially on ζ . Integrable models obtained via such deformations have been studied in [8–10] and [11–14] (see also the references in these papers). We demonstrate that in this way one can also recover hierarchies associated with derivative nonlinear Schrödinger equations (DNLS). Instead of working with deformations of a Lie algebra, we consider deformations of an associative algebra (which then, via the commutator, induce a Lie algebra structure). Although this somewhat restricts the possibilities, it allows some technical steps which greatly simplify the analysis (see also the remark in section 2). The problem of deformations of associative products also arises in

the context of compatible Poisson structures, which is of relevance in the theory of integrable systems too (see [15–17], in particular).

In this work, we derive moreover *functional representations* of DNLS hierarchies. These are generating equations, dependent on auxiliary parameters, which determine the hierarchy equations directly in terms of the relevant dependent variables (see also [18–26]). We make use of a method developed in [27], which will be recalled in section 3.1. Here, it is of relevance that AKNS hierarchies can be reformulated as ‘Gelfand–Dickey-type’ hierarchies (see (2.11)).

We actually derive functional representations for hierarchies with dependent variables in any (possibly noncommutative) associative algebra (see also [28–30], for example). Specialization to matrix algebras then covers cases of coupled systems of equations. These are also a possible source of new integrable equations. A further motivation is provided by the operator method [31–33] which associates with a (scalar) nonlinear equation an operator version and then a suitable map from solutions of the latter to solutions of the former. For this method it is prerequisite, of course, to find an ‘integrable’ generalization of the respective equation with dependent variable in a noncommutative associative algebra.

Section 2 shows how derivative NLS hierarchies can be expressed as AKNS-type hierarchies by taking the possibility of deformations in the sense mentioned above into account. In subsections we obtain matrix versions of hierarchies associated with three variants (Chen–Lee–Liu, Gerdjikov–Ivanov and Kaup–Newell) of DNLS equations in this way. Section 3 recalls some results from [27] which are then used to derive functional representations of the DNLS hierarchies. Section 4 contains some concluding remarks. Appendix A provides elementary material on deformations of products, as used in the main part of this work. As outlined in appendix B, in cases where it is possible to ‘undeform’ the product, contact with other formulations of the respective hierarchies can be established.

2. Derivative NLS hierarchies as AKNS-type hierarchies

Let (\mathcal{A}, \bullet) be an associative algebra (over a field \mathbb{K} of characteristic zero, typically \mathbb{R} or \mathbb{C}) with unit J . The product \bullet trivially extends to the algebra $\hat{\mathcal{A}} := \mathcal{A}[\zeta, \zeta^{-1}]$ of polynomials in an indeterminate ζ and formal power series in ζ^{-1} (with coefficients in \mathcal{A}). In the following, we consider an AKNS-type hierarchy

$$\partial_n(V) = [(\zeta^n V)_{\geq k}, V]_{\bullet} = -[(\zeta^n V)_{< k}, V]_{\bullet}, \quad n = 1, 2, \dots, \quad (2.1)$$

where $[X, Y]_{\bullet} := X \bullet Y - Y \bullet X$. Without restriction of generality we may take V of the form

$$V = u_0 + u_1 \zeta^{-1} + u_2 \zeta^{-2} + u_3 \zeta^{-3} + \dots \quad (2.2)$$

with $u_n \in \mathcal{A}$. Note that (2.1) is invariant under an additive shift of V by some constant element in the centre of $\hat{\mathcal{A}}$, e.g., any multiple of the unit element. As a consequence, there is a certain arbitrariness in the choice of u_0 in (2.2). For example, the original AKNS hierarchy (cf [3], for example) is obtained with an algebra of 2×2 matrices and $u_0 = \text{diag}(1, -1)$ (up to multiplication by $-\iota$), but we may choose $u_0 = \text{diag}(1, 0)$ as well. In this work we actually concentrate on examples based on algebras of 2×2 matrices, but with entries in an associative and typically noncommutative algebra³.

Equation (2.1) indeed defines a hierarchy, i.e. the flows mutually commute, if $\hat{\mathcal{A}}$ admits a direct sum decomposition $\hat{\mathcal{A}} = \hat{\mathcal{A}}_- \oplus \hat{\mathcal{A}}_+$, where $\hat{\mathcal{A}}_- := (\hat{\mathcal{A}})_{< k}$ and $\hat{\mathcal{A}}_+ := (\hat{\mathcal{A}})_{\geq k}$ are subalgebras, i.e.,

$$\hat{\mathcal{A}}_- \bullet \hat{\mathcal{A}}_- \subset \hat{\mathcal{A}}_-, \quad \hat{\mathcal{A}}_+ \bullet \hat{\mathcal{A}}_+ \subset \hat{\mathcal{A}}_+. \quad (2.3)$$

³ In the case of $N \times N$ matrices with $N > 2$, we should extend (2.1) to a multi-component version, see [6], for example.

This is the case if $k \in \{0, 1\}$. Let us now allow deformed associative products, with the indeterminate ζ as the deformation parameter.

$\mathbf{k} = 0$. Let $\bullet_{(0)}$ and $\bullet_{(-1)}$ be two compatible associative products in \mathcal{A} ([15–17], see also appendix A). Then, the new product

$$A \bullet B := A \bullet_{(0)} B + A \bullet_{(-1)} B \zeta \tag{2.4}$$

is associative for all ζ and, extended to $\hat{\mathcal{A}}$, satisfies

$$(X_{\geq 0} \bullet Y_{\geq 0})_{\geq 0} = X_{\geq 0} \bullet Y_{\geq 0}, \quad (X_{< 0} \bullet Y_{< 0})_{< 0} = X_{< 0} \bullet Y_{< 0} \tag{2.5}$$

for all $X, Y \in \hat{\mathcal{A}}$, so that the subalgebra conditions (2.3) are preserved. Any other ζ -dependence of the new product but the linear one would spoil one of these relations. Concerning our notation, assigning the index -1 to the second product is in accordance with an effective ζ -grading⁴.

$\mathbf{k} = 1$. Given two compatible associative products $\bullet_{(0)}$ and $\bullet_{(1)}$ in \mathcal{A} , the new product

$$A \bullet B := A \bullet_{(0)} B + A \bullet_{(1)} B \zeta^{-1} \tag{2.6}$$

is associative for all ζ and, extended to $\hat{\mathcal{A}}$, satisfies

$$(X_{\geq 1} \bullet Y_{\geq 1})_{\geq 1} = X_{\geq 1} \bullet Y_{\geq 1}, \quad (X_{< 1} \bullet Y_{< 1})_{< 1} = X_{< 1} \bullet Y_{< 1} \tag{2.7}$$

for all $X, Y \in \hat{\mathcal{A}}$, so that (2.3) is preserved. In fact, the ζ -dependence in (2.6) is the only one with this property.

In the following, we assume that V is given by a dressing

$$V = W \bullet P \bullet W^{\bullet-1} \tag{2.8}$$

where W is an invertible formal power series in ζ^{-1} with inverse $W^{\bullet-1}$ (with respect to the product \bullet) and $P \in \mathcal{A}$ is constant and idempotent, i.e.,

$$P \bullet P = P. \tag{2.9}$$

Then (2.8) leads to

$$V \bullet V = V. \tag{2.10}$$

Introducing $L := V \zeta$, this implies⁵ $L^{\bullet n} = \zeta^n V$, so that the AKNS-type hierarchy (2.1) can be equivalently expressed as

$$L_{t_n} = [(L^{\bullet n})_{\geq k}, L]_{\bullet} \quad n = 1, 2, \dots \tag{2.11}$$

This observation will be important in section 3.

Remark. In view of the structure of (2.1), it seems to be more natural and more general to replace the associative algebra \mathcal{A} by a Lie algebra which splits into a direct sum of Lie subalgebras. Of course, the commutator with respect to the product \bullet in \mathcal{A} induces a Lie algebra structure, but not every Lie algebra can be obtained in this way. Deformations of Lie algebras in the context of integrable systems have been studied in particular in [8–10] and [11–14]. The two classes of Lie algebras obtained from the deformations of associative algebras considered above are *quasigraded* of type $(1, 0)$, respectively $(0, 1)$, in the sense of the latter references. Although in (2.9) and (2.10) we do make explicit use of an *associative algebra structure*, the subsequent results in this section can also be obtained in a Lie algebraic framework, though with some more efforts⁶. The deeper reason for our choice of the associative algebra framework is the applicability of the methods developed in [27] to compute functional representations (see section 3). This choice is already required for the conversion of (2.1) into (2.11).

⁴ For example, taking this convention into account, the sum of all indices of each summand of (2.15) or (2.17) equals $n + 1$. This helps to keep account of the possible terms.

⁵ Here $L^{\bullet n}$ denotes the n -fold product $L \bullet L \cdots \bullet L$.

⁶ The methods developed in [11–14] can certainly also be used to derive the results of this section.

2.1. The case $k = 0$

Equation (2.10) leads to

$$u_0 \bullet_{(-1)} u_0 = 0, \quad (2.12)$$

$$\sum_{j=0}^n u_j \bullet_{(0)} u_{n-j} + \sum_{j=0}^{n+1} u_j \bullet_{(-1)} u_{n-j+1} = u_n \quad n = 0, 1, 2, \dots \quad (2.13)$$

The hierarchy equations (2.1) imply, in particular,

$$u_{0,x} = [u_0, u_2]_{(-1)}, \quad (2.14)$$

$$u_{n,x} = [u_0, u_{n+1}]_{(0)} + [u_1, u_n]_{(0)} + [u_0, u_{n+2}]_{(-1)} + [u_1, u_{n+1}]_{(-1)} \quad n = 1, 2, \dots \quad (2.15)$$

and

$$u_{0,t_n} = [u_0, u_{n+1}]_{(-1)}, \quad (2.16)$$

$$u_{1,t_n} = [u_0, u_{n+1}]_{(0)} + [u_0, u_{n+2}]_{(-1)} + [u_1, u_{n+1}]_{(-1)} \quad n = 1, 2, \dots \quad (2.17)$$

Combining (2.15) and (2.17) leads to

$$u_{1,t_n} = u_{n,x} - [u_1, u_n]_{(0)} \quad n = 1, 2, \dots \quad (2.18)$$

If we assume in addition that

$$u_0 = P \quad (2.19)$$

and

$$P \bullet_{(-1)} A = 0 = A \bullet_{(-1)} P \quad \forall A \in \mathcal{A}, \quad (2.20)$$

then (2.12), (2.13) for $n = 0$, (2.14) and (2.16) are satisfied (by use of (2.9)).

2.1.1. Chen–Lee–Liu DNLS hierarchy. Let \mathcal{A} be a matrix algebra. We define the product \bullet by⁷

$$A \bullet B := APB + A(I - P)B\zeta \quad (2.21)$$

with the unit matrix I and a fixed matrix P . For $\zeta = 1$, this is the usual matrix product. It is easily checked that the deformed product is indeed associative (see also appendix A). If $P^2 = P$, then (2.9) holds and (\mathcal{A}, \bullet) is unital with unit

$$J = P + (I - P)\zeta^{-1}. \quad (2.22)$$

Furthermore, (2.20) holds. Now we turn to a more concrete example by choosing

$$u_0 = P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (2.23)$$

Equation (2.13) with $n = 1$ is then solved by

$$u_1 = \begin{pmatrix} -qr & q \\ r & 0 \end{pmatrix}, \quad (2.24)$$

⁷ The corresponding Lie algebra version is a special case of the Lie algebra deformations considered in [12, 14], for example.

where q and r are $M \times N$, respectively $N \times M$ matrices, with entries from a possibly noncommutative unital algebra (over \mathbb{K}). Equation (2.13) with $n = 2$ leads to

$$u_2 = \begin{pmatrix} -(qr)^2 - qc - br & b \\ c & rq \end{pmatrix} \quad (2.25)$$

where the entries b, c still have to be determined. Now we use (2.15) with $n = 1$ to obtain

$$u_2 = \begin{pmatrix} qr_x - q_x r + (qr)^2 & q_x - qrq \\ -r_x - rqr & rq \end{pmatrix}. \quad (2.26)$$

Equation (2.18) for $n = 2$ then takes the form⁸

$$q_{t_2} - q_{xx} + 2q_x r q = 0, \quad r_{t_2} + r_{xx} + 2r q r_x = 0. \quad (2.27)$$

These are ‘noncommutative’ (e.g., matrix) versions of a system of coupled derivative nonlinear Schrödinger (DNLS) equations, generalizing the *Chen–Lee–Liu equation* [34–36].⁹ In the next step we obtain

$$u_3 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (2.28)$$

where

$$a := -qr_{xx} - q_{xx}r + q_x r_x + 2q_x r q r - 2qrq r_x + qdr, \quad (2.29)$$

$$b := q_{xx} - 2q_x r q - qd, \quad (2.30)$$

$$c := r_{xx} + 2r q r_x - dr, \quad (2.31)$$

$$d := r q_x - r_x q - (rq)^2. \quad (2.32)$$

Equation (2.18) for $n = 3$ then becomes the system

$$q_{t_3} - q_{xxx} + 3q_{xx} r q + 3q_x r q_x - 3q_x (rq)^2 = 0, \quad (2.33)$$

$$r_{t_3} - r_{xxx} - 3r q r_{xx} - 3r_x q r_x - 3(rq)^2 r_x = 0. \quad (2.34)$$

2.1.2. Gerdjikov–Ivanov DNLS hierarchy. Let \mathcal{A} be a matrix algebra, $\sigma \in \mathcal{A}$ such that

$$\sigma^2 = I, \quad (2.35)$$

and

$$A^{(-)} := \frac{1}{2}(A - \sigma A \sigma), \quad A^{(+)} := A - A^{(-)}, \quad (2.36)$$

for $A \in \mathcal{A}$. This provides us with a decomposition $\mathcal{A} = \mathcal{A}^{(+)} \oplus \mathcal{A}^{(-)}$. Then,

$$A \bullet B := (AB - A^{(-)} B^{(-)}) + A^{(-)} B^{(-)} \zeta \quad (2.37)$$

defines an associative deformation of the matrix product (see also appendix A)¹⁰. The unit matrix I is also a unit element with respect to the deformed product, hence $J = I$. Furthermore, $P := (I + \sigma)/2$ satisfies $P^{(-)} = 0$ and thus $P \bullet P = P^2 = P$.

⁸ These equations are of ‘Schrödinger type’ after replacing t_2 by ιt_2 with the imaginary unit ι . In the case $M = N$, the reductions $q = 1$ or $r = 1$, where 1 stands for a unit element, lead to noncommutative versions of the Burgers equation.

⁹ This includes the case of vector DNLS equations ($M = 1$ or $N = 1$). See also [37] and the references cited therein for the physical relevance of vector NLS equations.

¹⁰ For 2×2 matrices, the products $AB - A^{(-)} B^{(-)}$ and $A^{(-)} B^{(-)}$ appeared as examples in [15], in a different context.

Choosing

$$u_0 = P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.38)$$

(2.13) for $n = 1$ leads to

$$u_1 = \begin{pmatrix} -qr & q \\ r & rq \end{pmatrix}. \quad (2.39)$$

From (2.13) with $n = 2$, and (2.15), we find

$$u_2 = \begin{pmatrix} qr_x - q_x r - (qr)^2 & q_x \\ -r_x & rq_x - r_x q + (rq)^2 \end{pmatrix}. \quad (2.40)$$

According to (2.18), the first system of the hierarchy is

$$q_{t_2} - q_{xx} - 2qr_x q + 2(qr)^2 q = 0, \quad (2.41)$$

$$r_{t_2} + r_{xx} - 2rq_x r - 2r(qr)^2 = 0, \quad (2.42)$$

which generalizes the *Gerdjikov–Ivanov equation*, a derivative nonlinear Schrödinger equation with an additional potential term [28, 38–45].

2.2. The case $k = 1$

From (2.10), we obtain

$$u_0 \bullet_{(0)} u_0 = u_0, \quad (2.43)$$

$$\sum_{j=0}^n u_j \bullet_{(0)} u_{n-j} + \sum_{j=0}^{n-1} u_j \bullet_{(1)} u_{n-j-1} = u_n \quad n = 1, 2, \dots, \quad (2.44)$$

and the hierarchy equations (2.1) lead to

$$u_{n,x} = [u_0, u_{n+1}]_{(0)} + [u_0, u_n]_{(1)} \quad n = 0, 1, 2, \dots \quad (2.45)$$

and

$$u_{0,t_n} = [u_0, u_n]_{(0)} \quad n = 1, 2, \dots \quad (2.46)$$

Combined with (2.45), this yields

$$u_{0,t_{n+1}} - u_{n,x} + [u_0, u_n]_{(1)} = 0 \quad n = 0, 1, 2, \dots \quad (2.47)$$

2.2.1. Kaup–Newell hierarchy. Now we choose the product \bullet as in section 2.1.2, but with ζ replaced by ζ^{-1} .¹¹ Hence,

$$A \bullet_{(0)} B := AB - A^{(-)} B^{(-)} = A^{(+)} B^{(+)} + A^{(+)} B^{(-)} + A^{(-)} B^{(+)}, \quad (2.48)$$

$$A \bullet_{(1)} B := A^{(-)} B^{(-)}. \quad (2.49)$$

Furthermore, (2.45) and (2.46) split as follows:

$$u_{n,x}^{(+)} = [u_0^{(+)}, u_{n+1}^{(+)}] + [u_0^{(-)}, u_n^{(-)}], \quad (2.50)$$

¹¹ The choice of the product used in section 2.1.1 reproduces the results obtained there.

$$u_{n,x}^{(-)} = [u_0^{(+)}, u_{n+1}^{(-)}] + [u_0^{(-)}, u_{n+1}^{(+)}], \tag{2.51}$$

where $n = 0, 1, \dots$, and

$$u_{0,t_n}^{(+)} = [u_0^{(+)}, u_n^{(+)}], \tag{2.52}$$

$$u_{0,t_n}^{(-)} = [u_0^{(+)}, u_n^{(-)}] + [u_0^{(-)}, u_n^{(+)}] = u_{n-1,x}^{(-)}, \tag{2.53}$$

where $n = 1, 2, \dots$. Choosing moreover

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.54}$$

equation (2.43) and also (2.50) for $n = 0$ are solved by

$$u_0 = \begin{pmatrix} 1 & q \\ r & 0 \end{pmatrix}. \tag{2.55}$$

where q, r are $(M \times N, \text{ respectively } N \times M)$ matrices with entries from a (possibly noncommutative) associative algebra. From (2.44) we obtain $u_1 = u_0 \bullet_{(0)} u_1 + u_1 \bullet_{(0)} u_0 + u_0 \bullet_{(1)} u_0$. Together with (2.51) for $n = 0$, this entails

$$u_1 = \begin{pmatrix} -qr & q_x - 2qrq \\ -r_x - 2rqr & rq \end{pmatrix}. \tag{2.56}$$

Now (2.53) leads to

$$q_{t_2} = q_{xx} - 2(qrq)_x, \quad r_{t_2} = -r_{xx} - 2(rqr)_x \tag{2.57}$$

which generalizes the *Kaup–Newell* derivative NLS equation [46–49].

3. Functional representations of the derivative NLS hierarchies

After recalling a result from [27], we apply it to compute functional representations of the (matrix) DNLS hierarchies considered in section 2.

3.1. Preliminaries

In [27] we showed that an integrable hierarchy, from a class which includes (2.11), implies (and is typically equivalent to)

$$E(\lambda_2)_{-[\lambda_1]} \bullet E(\lambda_1) = E(\lambda_1)_{-[\lambda_2]} \bullet E(\lambda_2), \tag{3.1}$$

where λ_1 and λ_2 are indeterminates and

$$E(\lambda) = J + \sum_{n \geq 1} \lambda^n E_n \tag{3.2}$$

is a formal power series with coefficients

$$E_n = (\tilde{E}_n)_+ \quad n = 1, 2, \dots, \tag{3.3}$$

recursively determined via

$$\tilde{E}_1 = -L, \quad \tilde{E}_{n+1} = (\tilde{E}_n)_- \bullet L \quad n = 1, 2, \dots \tag{3.4}$$

Furthermore, we used the following notation (see also [20, 27, 50, 51], for example). For F dependent on $\mathbf{t} := (t_1, t_2, t_3, \dots)$, we define

$$F_{[\lambda]}(\mathbf{t}) := F(\mathbf{t} + [\lambda]) = \sum_{n \geq 0} \lambda^n \chi_n(F), \quad F_{-[\lambda]}(\mathbf{t}) := F(\mathbf{t} - [\lambda]) \tag{3.5}$$

(as formal power series in λ) with

$$[\lambda] := (\lambda, \lambda^2/2, \lambda^3/3, \dots) \quad (3.6)$$

and

$$\chi_n := \mathbf{p}_n(\tilde{\partial}), \quad \tilde{\partial} := (\partial_{t_1}, \partial_{t_2}/2, \partial_{t_3}/3, \dots), \quad (3.7)$$

where $\mathbf{p}_n, n = 0, 1, 2, \dots$, are the elementary Schur polynomials (see [20, 52, 53], for example).

It is helpful to express (3.1) in terms of

$$\hat{E}(\lambda) := E(\lambda)_{[\lambda]}, \quad (3.8)$$

so that it takes the form

$$\hat{E}(\lambda_2) \bullet \hat{E}(\lambda_1)_{[\lambda_2]} = \hat{E}(\lambda_1) \bullet \hat{E}(\lambda_2)_{[\lambda_1]}. \quad (3.9)$$

Remark. (3.1) (or (3.9)) should be regarded as a *finite* (i.e. group) analogue of an *infinitesimal* (i.e., Lie algebraic) zero curvature equation. $E(\lambda)$ (or $\hat{E}(\lambda)$) lies in the group of invertible elements of the algebra $\hat{\mathcal{A}}$. In certain cases it should be possible to regard this group as a (formal) Lie group of a (loop) Lie algebra $\hat{\mathcal{G}} = \hat{\mathcal{G}}_+ \oplus \hat{\mathcal{G}}_-$. More precisely, $E(\lambda)$ would then be an element of the group generated by $\hat{\mathcal{G}}_+$. However, such a Lie algebraic relation may impose unnecessary and inconvenient restrictions on $E(\lambda)$. In the subsequent calculations it is not at all required.

3.2. The case $k = 0$

We have

$$E(\lambda) = J - (u_0\zeta + u_1)\lambda + \sum_{n \geq 2} (\tilde{E}_n)_{\geq 0} \lambda^n \quad (3.10)$$

and $E_n = (\tilde{E}_n)_{\geq 0}$ is *linear* in ζ for $n = 2, 3, \dots$. This follows from

$$\begin{aligned} (\tilde{E}_{n+1})_{\geq 0} &= ((\tilde{E}_n)_{<0} \bullet (u_0\zeta + u_1))_{\geq 0} \\ &= \text{res}(\tilde{E}_n) \bullet_{(-1)} u_0\zeta + \text{res}(\tilde{E}_n) \bullet_{(0)} u_0 + \text{res}(\tilde{E}_n\zeta) \bullet_{(-1)} u_0 + \text{res}(\tilde{E}_n) \bullet_{(-1)} u_1, \end{aligned} \quad (3.11)$$

where $\text{res}(X)$ takes the coefficient of the ζ^{-1} part of X . Let us write

$$\hat{E}(\lambda) = J - (v(\lambda)\zeta + w(\lambda))\lambda \quad (3.12)$$

where $v(\lambda) = \sum_{n \geq 0} v_n \lambda^n$ and $w(\lambda) = \sum_{n \geq 0} w_n \lambda^n$ are (formal) power series in λ with $v_0 = u_0$ and $w_0 = u_1$. Now (3.9) splits into the four equations

$$v(\lambda_2) \bullet_{(-1)} v(\lambda_1)_{[\lambda_2]} = v(\lambda_1) \bullet_{(-1)} v(\lambda_2)_{[\lambda_1]}, \quad (3.13)$$

$$\begin{aligned} v(\lambda_2) \bullet_{(0)} v(\lambda_1)_{[\lambda_2]} + v(\lambda_2) \bullet_{(-1)} w(\lambda_1)_{[\lambda_2]} + w(\lambda_2) \bullet_{(-1)} v(\lambda_1)_{[\lambda_2]} &= v(\lambda_1) \bullet_{(0)} v(\lambda_2)_{[\lambda_1]} \\ &+ v(\lambda_1) \bullet_{(-1)} w(\lambda_2)_{[\lambda_1]} + w(\lambda_1) \bullet_{(-1)} v(\lambda_2)_{[\lambda_1]}, \end{aligned} \quad (3.14)$$

$$\begin{aligned} \lambda_1^{-1} (v(\lambda_2)_{[\lambda_1]} - v(\lambda_2)) - \lambda_2^{-1} (v(\lambda_1)_{[\lambda_2]} - v(\lambda_1)) &= v(\lambda_1) \bullet_{(0)} w(\lambda_2)_{[\lambda_1]} \\ &- v(\lambda_2) \bullet_{(0)} w(\lambda_1)_{[\lambda_2]} + w(\lambda_1) \bullet_{(0)} v(\lambda_2)_{[\lambda_1]} - w(\lambda_2) \bullet_{(0)} v(\lambda_1)_{[\lambda_2]} \\ &+ w(\lambda_1) \bullet_{(-1)} w(\lambda_2)_{[\lambda_1]} - w(\lambda_2) \bullet_{(-1)} w(\lambda_1)_{[\lambda_2]}, \end{aligned} \quad (3.15)$$

$$\begin{aligned} \lambda_1^{-1} (w(\lambda_2)_{[\lambda_1]} - w(\lambda_2)) - \lambda_2^{-1} (w(\lambda_1)_{[\lambda_2]} - w(\lambda_1)) \\ = w(\lambda_1) \bullet_{(0)} w(\lambda_2)_{[\lambda_1]} - w(\lambda_2) \bullet_{(0)} w(\lambda_1)_{[\lambda_2]}. \end{aligned} \quad (3.16)$$

Assuming (2.19) and (2.20), equation (3.11) shows that

$$v(\lambda) = P, \tag{3.17}$$

and the first two equations are identically satisfied. Furthermore, the third equation reduces to

$$P \bullet_{(0)} (w(\lambda_2)_{[\lambda_1]} - w(\lambda_1)_{[\lambda_2]}) + (w(\lambda_1) - w(\lambda_2)) \bullet_{(0)} P = w(\lambda_2) \bullet_{(-1)} w(\lambda_1)_{[\lambda_2]} - w(\lambda_1) \bullet_{(-1)} w(\lambda_2)_{[\lambda_1]}. \tag{3.18}$$

In the limit $\lambda_2 \rightarrow 0$, equations (3.16) and (3.18) lead to

$$\lambda^{-1}(w_{0,[\lambda]} - w_0) - w(\lambda)_x = w(\lambda) \bullet_{(0)} w_{0,[\lambda]} - w_0 \bullet_{(0)} w(\lambda), \tag{3.19}$$

respectively

$$P \bullet_{(0)} (w_{0,[\lambda]} - w(\lambda)) + (w(\lambda) - w_0) \bullet_{(0)} P = w_0 \bullet_{(-1)} w(\lambda) - w(\lambda) \bullet_{(-1)} w_{0,[\lambda]}. \tag{3.20}$$

3.2.1. *Functional representation of the Chen–Lee–Liu DNLS hierarchy.* We choose \mathcal{A} , the product \bullet and P as in section 2.1.1. Let us now turn to the derivation of a corresponding functional representation of the hierarchy. We write

$$w(\lambda) = \begin{pmatrix} p(\lambda) & q(\lambda) \\ r(\lambda) & s(\lambda) \end{pmatrix}, \tag{3.21}$$

where the entries are power series in λ . In this case, (3.11) reads

$$E_{n+1} = (\tilde{E}_{n+1})_{\geq 0} = \text{res}(\tilde{E}_n)P + \text{res}(\tilde{E}_n)(I - P)u_1. \tag{3.22}$$

Since $(u_1)_{22} = 0$ according to (2.24), we easily deduce from this formula that

$$s(\lambda) = 0. \tag{3.23}$$

Now (3.16) and (3.18) are turned into the following equations:

$$\lambda_2^{-1}(p(\lambda_1)_{[\lambda_2]} - p(\lambda_1)) - \lambda_1^{-1}(p(\lambda_2)_{[\lambda_1]} - p(\lambda_2)) = p(\lambda_2)p(\lambda_1)_{[\lambda_2]} - p(\lambda_1)p(\lambda_2)_{[\lambda_1]}, \tag{3.24}$$

$$\lambda_2^{-1}(q(\lambda_1)_{[\lambda_2]} - q(\lambda_1)) - \lambda_1^{-1}(q(\lambda_2)_{[\lambda_1]} - q(\lambda_2)) = p(\lambda_2)q(\lambda_1)_{[\lambda_2]} - p(\lambda_1)q(\lambda_2)_{[\lambda_1]}, \tag{3.25}$$

$$\lambda_2^{-1}(r(\lambda_1)_{[\lambda_2]} - r(\lambda_1)) - \lambda_1^{-1}(r(\lambda_2)_{[\lambda_1]} - r(\lambda_2)) = r(\lambda_2)p(\lambda_1)_{[\lambda_2]} - r(\lambda_1)p(\lambda_2)_{[\lambda_1]}, \tag{3.26}$$

$$r(\lambda_1)q(\lambda_2)_{[\lambda_1]} - r(\lambda_2)q(\lambda_1)_{[\lambda_2]} = 0, \tag{3.27}$$

and

$$p(\lambda_1)_{[\lambda_2]} - p(\lambda_1) - p(\lambda_2)_{[\lambda_1]} + p(\lambda_2) = q(\lambda_1)r(\lambda_2)_{[\lambda_1]} - q(\lambda_2)r(\lambda_1)_{[\lambda_2]}, \tag{3.28}$$

$$q(\lambda_1)_{[\lambda_2]} - q(\lambda_2)_{[\lambda_1]} = 0, \tag{3.29}$$

$$r(\lambda_2) - r(\lambda_1) = 0. \tag{3.30}$$

As a consequence of (3.29) and (3.30), we have

$$q(\lambda) = q_{[\lambda]}, \quad r(\lambda) = r, \tag{3.31}$$

and (3.27) is automatically satisfied. Now (3.28) becomes

$$p(\lambda_1)_{[\lambda_2]} - p(\lambda_1) - p(\lambda_2)_{[\lambda_1]} + p(\lambda_2) = (qr)_{[\lambda_1]} - (qr)_{[\lambda_2]}. \tag{3.32}$$

In the limit $\lambda_2 \rightarrow 0$, this yields $-p_{0,[\lambda]} + p_0 = (qr)_{[\lambda]} - qr$, which is solved by $p_0 = -qr$, in accordance with the upper left entry of u_1 in (2.24). The coefficient of the $\lambda_1^n \lambda_2^m$ term, $m, n \in \mathbb{N}$, of the above equation reads $\chi_n(p_m) = \chi_m(p_n)$. Hence, there is a p such that

$$p_n = \chi_n(p) \quad n = 1, 2, \dots, \tag{3.33}$$

and consequently

$$p(\lambda) = p_{[\lambda]} - p - qr. \quad (3.34)$$

Taking the limit $\lambda_2 \rightarrow 0$ of (3.24), (3.25) and (3.26), and using (3.31), we obtain

$$p(\lambda)_x = -\lambda^{-1}((qr)_{[\lambda]} - qr) + p(\lambda)(qr)_{[\lambda]} - qrp(\lambda), \quad (3.35)$$

$$q_{[\lambda],x} = \lambda^{-1}(q_{[\lambda]} - q) - (qr + p(\lambda))q_{[\lambda]}, \quad (3.36)$$

$$r_x = \lambda^{-1}(r_{[\lambda]} - r) + r(p(\lambda) + (qr)_{[\lambda]}). \quad (3.37)$$

Multiplying (3.36) by r from the right and (3.37) by $q_{[\lambda]}$ from the left, adding the results and using (3.34), we get

$$(q_{[\lambda]}r)_x = (\lambda^{-1} + q_{[\lambda]}r)((qr)_{[\lambda]} - qr) + [q_{[\lambda]}r, p_{[\lambda]} - p]. \quad (3.38)$$

With the help of (3.34) we rewrite (3.35) as follows:

$$p(\lambda)_x = -(\lambda^{-1} - p(\lambda))((qr)_{[\lambda]} - qr) + [p(\lambda), p_{[\lambda]} - p]. \quad (3.39)$$

Adding the last two equations provides us with

$$(p(\lambda) + q_{[\lambda]}r)_x = (p(\lambda) + q_{[\lambda]}r)((qr)_{[\lambda]} - qr) + [p(\lambda) + q_{[\lambda]}r, p_{[\lambda]} - p]. \quad (3.40)$$

Expanding in powers of λ , using $p_0 = -qr$ and setting possible integration constants to zero¹² leads to

$$p(\lambda) = -q_{[\lambda]}r. \quad (3.41)$$

Inserting this in (3.36) and (3.37), we obtain the following functional representation of the hierarchy,

$$(q - q_{-[\lambda]})(\lambda^{-1} + r_{-[\lambda]}q) = q_x, \quad (3.42)$$

$$(\lambda^{-1} + rq_{[\lambda]})(r_{[\lambda]} - r) = r_x. \quad (3.43)$$

By differentiation with respect to λ , we recover the corresponding equations in [24], obtained in the case of commuting variables.

Remark. Expanding (3.24) and using (3.33), the coefficients of $\lambda_1^m \lambda_2^n$, $m, n \geq 1$ entail

$$\chi_{n+1}(\chi_m(p)) - \chi_{m+1}(\chi_n(p)) = \sum_{j=1}^n \chi_j(p) \chi_{n-j}(\chi_m(p)) - \sum_{j=1}^m \chi_j(p) \chi_{m-j}(\chi_n(p)). \quad (3.44)$$

This system is equivalent to the potential KP hierarchy (see section 3.3 in [27]). Since (3.34) implies $p_x = -qr_x$, it follows that

$$p = - \int q_x r \, dx \quad (3.45)$$

solves the potential KP hierarchy if q and r satisfy the above Chen–Lee–Liu DNLS hierarchy.

¹² This would be enforced by imposing an auxiliary boundary condition like $w(\lambda) \rightarrow 0$ as $x \rightarrow \pm\infty$.

3.2.2. *Functional representation of the Gerdjikov–Ivanov DNLS hierarchy.* We choose \mathcal{A} , the product \bullet and P as in section 2.1.2. In order to determine a functional representation of the hierarchy, we write again

$$w(\lambda) = \begin{pmatrix} p(\lambda) & q(\lambda) \\ r(\lambda) & s(\lambda) \end{pmatrix} \tag{3.46}$$

where the entries are (formal) power series in λ , and obtain from (3.18), as in section 3.2.1,

$$q(\lambda) = q_{[\lambda]}, \quad r(\lambda) = r, \quad p(\lambda) = p_{[\lambda]} - p - qr. \tag{3.47}$$

From (3.16) we obtain the system

$$\lambda_1^{-1}(p(\lambda_2)_{[\lambda_1]} - p(\lambda_2)) - \lambda_2^{-1}(p(\lambda_1)_{[\lambda_2]} - p(\lambda_1)) = p(\lambda_1)p(\lambda_2)_{[\lambda_1]} - p(\lambda_2)p(\lambda_1)_{[\lambda_2]}, \tag{3.48}$$

$$\lambda_1^{-1}(s(\lambda_2)_{[\lambda_1]} - s(\lambda_2)) - \lambda_2^{-1}(s(\lambda_1)_{[\lambda_2]} - s(\lambda_1)) = s(\lambda_1)s(\lambda_2)_{[\lambda_1]} - s(\lambda_2)s(\lambda_1)_{[\lambda_2]}, \tag{3.49}$$

$$\begin{aligned} \lambda_1^{-1}(q_{[\lambda_1+\lambda_2]} - q_{[\lambda_2]}) - \lambda_2^{-1}(q_{[\lambda_1+\lambda_2]} - q_{[\lambda_1]}) &= (p(\lambda_1) - p(\lambda_2))q_{[\lambda_1+\lambda_2]} \\ &+ q_{[\lambda_1]}s(\lambda_2)_{[\lambda_1]} - q_{[\lambda_2]}s(\lambda_1)_{[\lambda_2]}, \end{aligned} \tag{3.50}$$

and

$$\lambda_1^{-1}(r_{[\lambda_1]} - r) - \lambda_2^{-1}(r_{[\lambda_2]} - r) = r(p(\lambda_2)_{[\lambda_1]} - p(\lambda_1)_{[\lambda_2]}) + s(\lambda_1)r_{[\lambda_1]} - s(\lambda_2)r_{[\lambda_2]}. \tag{3.51}$$

The $\lambda_1 \rightarrow 0$ limit of these equations leads to

$$(\lambda^{-1} - p(\lambda))_x = (\lambda^{-1} - p(\lambda))((qr)_{[\lambda]} - qr) + [\lambda^{-1} - p(\lambda), p_{[\lambda]} - p], \tag{3.52}$$

$$s(\lambda)_x = \lambda^{-1}[(rq)_{[\lambda]} - rq] + rqs(\lambda) - s(\lambda)(rq)_{[\lambda]}, \tag{3.53}$$

and

$$q_{[\lambda],x} = \lambda^{-1}(q_{[\lambda]} - q) - (p(\lambda) + (qr)_{[\lambda]} + qr)q_{[\lambda]} + qs(\lambda), \tag{3.54}$$

$$r_x = \lambda^{-1}(r_{[\lambda]} - r) + r(p(\lambda) + (qr)_{[\lambda]} + qr) - s(\lambda)r_{[\lambda]}. \tag{3.55}$$

Here, we made use of $s_0 = rq$, $p_0 = -qr$ (see (2.39)) and (3.47). Multiplying (3.54) by r from the left and (3.55) by $q_{[\lambda]}$ from the right, and adding the results, we get

$$(rq_{[\lambda]})_x = \lambda^{-1}[(rq)_{[\lambda]} - rq] + rqs(\lambda) - s(\lambda)(rq)_{[\lambda]}. \tag{3.56}$$

Comparing this with (3.53), we obtain

$$s(\lambda) = rq_{[\lambda]}, \tag{3.57}$$

setting a constant of integration to zero. Now (3.54) and (3.55) take the form

$$q_{[\lambda],x} = \lambda^{-1}(q_{[\lambda]} - q) - (p(\lambda) + (qr)_{[\lambda]})q_{[\lambda]}, \tag{3.58}$$

$$r_x = \lambda^{-1}(r_{[\lambda]} - r) + r(p(\lambda) + qr). \tag{3.59}$$

Multiplying the first equation by r from the right and the second by $q_{[\lambda]}$ from the left, adding the results and using the last of the relations (3.47) leads to

$$(\lambda^{-1} - q_{[\lambda]}r)_x = -((qr)_{[\lambda]} - qr)(\lambda^{-1} - q_{[\lambda]}r) + [\lambda^{-1} - q_{[\lambda]}r, p_{[\lambda]} - p]. \tag{3.60}$$

We multiply this in turn by $\lambda^{-1} - p(\lambda)$ from the left, and (3.52) by $\lambda^{-1} - q_{[\lambda]}r$ from the right, and add the results to obtain

$$[(\lambda^{-1} - p(\lambda))(\lambda^{-1} - q_{[\lambda]}r)]_x = [(\lambda^{-1} - p(\lambda))(\lambda^{-1} - q_{[\lambda]}r), p_{[\lambda]} - p]. \tag{3.61}$$

Integrating this order by order in λ leads to

$$(1 - \lambda p(\lambda))(1 - \lambda q_{[\lambda]}r) = 1, \quad (3.62)$$

or equivalently

$$p(\lambda) = -q_{[\lambda]}r(1 - \lambda q_{[\lambda]}r)^{-1} = -(1 - \lambda q_{[\lambda]}r)^{-1}q_{[\lambda]}r. \quad (3.63)$$

Inserting this expression in (3.58) and (3.59) yields the following functional form of the hierarchy:

$$\lambda^{-1}(q - q_{-[\lambda]}) - q(r - r_{-[\lambda]}(1 - \lambda q r_{-[\lambda]})^{-1})q = q_x, \quad (3.64)$$

$$\lambda^{-1}(r_{[\lambda]} - r) + r(q - (1 - \lambda q_{[\lambda]}r)^{-1}q_{[\lambda]}r) = r_x. \quad (3.65)$$

Again, we should stress that q and r may be $M \times N$ and $N \times M$ matrices of functions (or, more generally, matrices with elements of any (noncommutative) associative algebra), where $M, N \in \mathbb{N}$.

Remark. In the same way as in section 3.2.1 (see the remark there), we find that

$$p = - \int (q_x r + (qr)^2) dx \quad (3.66)$$

satisfies the potential KP hierarchy as a consequence of the above Gerdjikov–Ivanov DNLS hierarchy.

3.3. The case $k = 1$

According to (3.3) and (3.4), we have

$$E_0 = J, \quad E_1 = -L_{\geq 1} = -u_0 \zeta, \quad E_2 = ((\tilde{E}_1)_{<1} \bullet L)_{\geq 1} = -u_1 \bullet_{(0)} u_0 \zeta, \quad (3.67)$$

and

$$E_{n+1} = ((\tilde{E}_n)_{<1} \bullet L)_{\geq 1} = ((\tilde{E}_n)_{<1} \bullet u_0 \zeta)_{\geq 1} = \text{res}(\tilde{E}_n \zeta^{-1}) \bullet_{(0)} u_0 \zeta \quad n = 1, 2, \dots \quad (3.68)$$

Hence,

$$\hat{E}(\lambda) = E(\lambda)_{[\lambda]} = J - \lambda w(\lambda) \zeta, \quad w(\lambda) = u_0 + \lambda u_1 \bullet_{(0)} u_0 + \dots, \quad (3.69)$$

where $w(\lambda)$ does not depend on ζ . Now (3.9) yields the two equations

$$w(\lambda_2) \bullet_{(0)} w(\lambda_1)_{[\lambda_2]} - w(\lambda_1) \bullet_{(0)} w(\lambda_2)_{[\lambda_1]} = 0, \quad (3.70)$$

and

$$\lambda_2^{-1}(w(\lambda_1)_{[\lambda_2]} - w(\lambda_1)) - \lambda_1^{-1}(w(\lambda_2)_{[\lambda_1]} - w(\lambda_2)) = w(\lambda_2) \bullet_{(1)} w(\lambda_1)_{[\lambda_2]} - w(\lambda_1) \bullet_{(1)} w(\lambda_2)_{[\lambda_1]}. \quad (3.71)$$

In the limit $\lambda_2 \rightarrow 0$, this becomes

$$u_0 \bullet_{(0)} w(\lambda) = w(\lambda) \bullet_{(0)} u_{0,[\lambda]} \quad (3.72)$$

and

$$w(\lambda)_x - \lambda^{-1}(u_{0,[\lambda]} - u_0) = u_0 \bullet_{(1)} w(\lambda) - w(\lambda) \bullet_{(1)} u_{0,[\lambda]}. \quad (3.73)$$

3.3.1. *Functional representation of the Kaup–Newell hierarchy.* We choose \mathcal{A} , the product \bullet and P as in section 2.2.1. Let us write

$$w(\lambda) = \begin{pmatrix} p(\lambda) & q(\lambda) \\ r(\lambda) & s(\lambda) \end{pmatrix} \tag{3.74}$$

where the entries are formal power series in λ . Since $(u_0)_{22} = 0$ and $E_{n+1} = \text{res}(\tilde{E}_n \zeta^{-1})_{\bullet(0)} u_0$, we have

$$s(\lambda) = 0. \tag{3.75}$$

Furthermore, $q_0 = q, r_0 = r$ and $p_0 = 1$ according to (2.55). From (3.70) and (3.71) we obtain

$$p(\lambda_2)p(\lambda_1)_{[\lambda_2]} - p(\lambda_1)p(\lambda_2)_{[\lambda_1]} = 0, \tag{3.76}$$

$$p(\lambda_2)q(\lambda_1)_{[\lambda_2]} - p(\lambda_1)q(\lambda_2)_{[\lambda_1]} = 0, \tag{3.77}$$

$$r(\lambda_2)p(\lambda_1)_{[\lambda_2]} - r(\lambda_1)p(\lambda_2)_{[\lambda_1]} = 0, \tag{3.78}$$

and

$$\lambda_2^{-1}(p(\lambda_1)_{[\lambda_2]} - p(\lambda_1)) - \lambda_1^{-1}(p(\lambda_2)_{[\lambda_1]} - p(\lambda_2)) = q(\lambda_2)r(\lambda_1)_{[\lambda_2]} - q(\lambda_1)r(\lambda_2)_{[\lambda_1]}, \tag{3.79}$$

$$\lambda_2^{-1}(q(\lambda_1)_{[\lambda_2]} - q(\lambda_1)) - \lambda_1^{-1}(q(\lambda_2)_{[\lambda_1]} - q(\lambda_2)) = 0, \tag{3.80}$$

$$\lambda_2^{-1}(r(\lambda_1)_{[\lambda_2]} - r(\lambda_1)) - \lambda_1^{-1}(r(\lambda_2)_{[\lambda_1]} - r(\lambda_2)) = 0, \tag{3.81}$$

$$r(\lambda_2)q(\lambda_1)_{[\lambda_2]} - r(\lambda_1)q(\lambda_2)_{[\lambda_1]} = 0, \tag{3.82}$$

respectively. Equation (3.76) (with $p_0 = 1$) is solved by

$$p(\lambda) = f f_{[\lambda]}^{-1} \tag{3.83}$$

with some invertible f . Inserting this in (3.77), we obtain

$$(f^{-1}q(\lambda_2))_{[\lambda_1]} = (f^{-1}q(\lambda_1))_{[\lambda_2]}, \tag{3.84}$$

which implies

$$f^{-1}q(\lambda) = \tilde{q}_{[\lambda]} \tag{3.85}$$

with some \tilde{q} which is determined by taking the limit $\lambda \rightarrow 0$,

$$\tilde{q} = f^{-1}q. \tag{3.86}$$

Inserting (3.83) in (3.78), we obtain $r(\lambda_2)f_{[\lambda_2]} = r(\lambda_1)f_{[\lambda_1]}$ and thus

$$r(\lambda) = \tilde{r} f_{[\lambda]}^{-1} \quad \text{with} \quad \tilde{r} := r f. \tag{3.87}$$

As a consequence, (3.82) is automatically satisfied, and (3.79) takes the form

$$\lambda_2^{-1}(f^{-1}f_{[\lambda_2]} - (f^{-1}f_{[\lambda_2]})_{[\lambda_1]}) - \lambda_1^{-1}(f^{-1}f_{[\lambda_1]} - (f^{-1}f_{[\lambda_1]})_{[\lambda_2]}) = (\tilde{q}\tilde{r})_{[\lambda_2]} - (\tilde{q}\tilde{r})_{[\lambda_1]}. \tag{3.88}$$

In the limit $\lambda_2 \rightarrow 0$, this yields

$$f^{-1}f_x - (f^{-1}f_x)_{[\lambda]} = \tilde{q}\tilde{r} - (\tilde{q}\tilde{r})_{[\lambda]}, \tag{3.89}$$

and thus

$$f^{-1}f_x = \tilde{q}\tilde{r}, \tag{3.90}$$

up to addition of a constant, which we set to zero. The last equation is equivalent to $p_1 = -qr$. Expansion of (3.88) leads to

$$\chi_n(f^{-1}\chi_{m+1}(f)) = \chi_m(f^{-1}\chi_{n+1}(f)) \quad m, n = 1, 2, \dots, \quad (3.91)$$

which states that f has to solve the mKP hierarchy. Indeed, the last equation is a formulation of the mKP hierarchy (see [27]). It implies the existence of a potential ϕ such that

$$\chi_{n+1}(f) = -f\chi_n(\phi) \quad n = 1, 2, \dots, \quad (3.92)$$

which, in functional form, reads

$$f_{[\lambda]} = f + \lambda f_x - \lambda f(\phi_{[\lambda]} - \phi). \quad (3.93)$$

Hence,

$$f^{-1}f_{[\lambda]} = 1 + \lambda(\tilde{q}\tilde{r} - \phi_{[\lambda]} + \phi). \quad (3.94)$$

Using (3.90) and the last equation, we find

$$\begin{aligned} (f^{-1}f_{[\lambda]})_x &= -f^{-1}f_x f^{-1}f_{[\lambda]} + f^{-1}f_{[\lambda]}(f^{-1}f_x)_{[\lambda]} \\ &= (f^{-1}f_{[\lambda]})((\tilde{q}\tilde{r})_{[\lambda]} - \tilde{q}\tilde{r}) + [f^{-1}f_{[\lambda]}, \tilde{q}\tilde{r}] \\ &= (f^{-1}f_{[\lambda]})((\tilde{q}\tilde{r})_{[\lambda]} - \tilde{q}\tilde{r}) + [f^{-1}f_{[\lambda]}, \phi_{[\lambda]} - \phi]. \end{aligned} \quad (3.95)$$

Furthermore, the $\lambda_2 \rightarrow 0$ limits of (3.80) and (3.81) are

$$q(\lambda)_x = \lambda^{-1}(q_{[\lambda]} - q), \quad r(\lambda)_x = \lambda^{-1}(r_{[\lambda]} - r). \quad (3.96)$$

Conversely, these equations imply (3.80) and (3.81), since the x -derivatives of the latter are satisfied as a consequence of (3.96). Using (3.90) and (3.94), (3.96) becomes

$$\tilde{q}_{[\lambda],x} = \frac{1}{\lambda}(\tilde{q}_{[\lambda]} - \tilde{q}) - (\phi_{[\lambda]} - \phi)\tilde{q}_{[\lambda]}, \quad (3.97)$$

$$\tilde{r}_x = \frac{1}{\lambda}(\tilde{r}_{[\lambda]} - \tilde{r}) + \tilde{r}((\tilde{q}\tilde{r})_{[\lambda]} - \tilde{q}\tilde{r}) + \tilde{r}(\phi_{[\lambda]} - \phi). \quad (3.98)$$

Multiplying the first equation by \tilde{r} from the right, the second by $\tilde{q}_{[\lambda]}$ from the left and adding the results, we obtain

$$(\lambda^{-1} + \tilde{q}_{[\lambda]}\tilde{r})_x = (\lambda^{-1} + \tilde{q}_{[\lambda]}\tilde{r})((\tilde{q}\tilde{r})_{[\lambda]} - \tilde{q}\tilde{r}) + [\tilde{q}_{[\lambda]}\tilde{r}, \phi_{[\lambda]} - \phi]. \quad (3.99)$$

Now we multiply (3.95) by λ^{-1} and subtract the resulting equation from the preceding one to get

$$z(\lambda)_x = z(\lambda)((\tilde{q}\tilde{r})_{[\lambda]} - \tilde{q}\tilde{r}) + [z(\lambda), \phi_{[\lambda]} - \phi] \quad (3.100)$$

with the following power series in λ :

$$z(\lambda) := \lambda^{-1}(1 - f^{-1}f_{[\lambda]}) + \tilde{q}_{[\lambda]}\tilde{r}. \quad (3.101)$$

Expanding in powers of λ , a simple induction argument (using (3.90) and setting x -integration constants to zero) now leads to $z(\lambda) = 0$ and thus

$$f^{-1}f_{[\lambda]} = 1 + \lambda\tilde{q}_{[\lambda]}\tilde{r}, \quad (3.102)$$

which, with the help of (3.85) and (3.87), can be written in the form¹³

$$p(\lambda) = 1 - \lambda q(\lambda)r(\lambda). \quad (3.103)$$

From (3.85) and (3.87), using (3.83), we get

$$q(\lambda) = p(\lambda)q_{[\lambda]} = (1 - \lambda q(\lambda)r(\lambda))q_{[\lambda]}, \quad r(\lambda) = rp(\lambda) = r(1 - \lambda q(\lambda)r(\lambda)). \quad (3.104)$$

¹³ In the case of commuting variables, this formula can be derived in a much simpler way.

Introducing potentials via

$$q = Q_x, \quad r = R_x, \tag{3.105}$$

equations (3.96) can be integrated to give

$$q(\lambda) = \lambda^{-1}(Q_{[\lambda]} - Q), \quad r(\lambda) = \lambda^{-1}(R_{[\lambda]} - R). \tag{3.106}$$

Inserting this in (3.104), we arrive at the following functional representation of the hierarchy:

$$(Q - Q_{-[\lambda]})(1 + (R - R_{-[\lambda]})Q_x) = \lambda Q_x, \tag{3.107}$$

$$(1 + R_x(Q_{[\lambda]} - Q))(R_{[\lambda]} - R) = \lambda R_x. \tag{3.108}$$

Expansion in powers of λ leads, in lowest order, to

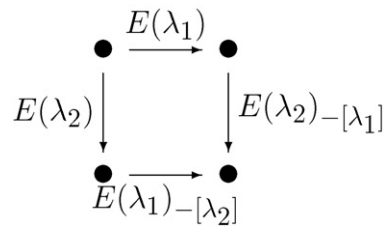
$$Q_{t_2} = Q_{xx} - 2Q_x R_x Q_x, \quad R_{t_2} = -R_{xx} - 2R_x Q_x R_x, \tag{3.109}$$

which is the potential form of (2.57).

4. Conclusions

Via deformations of associative products, we expressed derivative NLS hierarchies in the form of AKNS hierarchies and then also as ‘Gelfand–Dickey-type’ hierarchies (in the sense of (2.11)). We then applied a method to compute functional representations of ‘Gelfand–Dickey-type’ integrable hierarchies to the derivative NLS hierarchies. For the Chen–Lee–Liu DNLS hierarchy, we recovered corresponding formulae obtained previously by Vekslerchik [24]. The functional representations obtained in the case of the other derivative NLS hierarchies did not yet appear in the literature, according to our knowledge. In any case, the generalizations to noncommuting dependent variables appear to be new.

The results of [27] and the present work also demonstrate that certain properties of integrable hierarchies are surprisingly easily obtained from the formulation in terms of $E(\lambda)$, which has to solve the zero curvature conditions in the form (3.1). Regarding $E(\lambda)$ as a parallel transport operator taking objects at \mathbf{t} to objects at $\mathbf{t} - [\lambda]$, equation (3.1) attains the interpretation of a ‘discrete’ zero curvature condition¹⁴, which is depicted in the following (commutative) diagram:



Equation (3.1) has the following gauge invariance:

$$E(\lambda) \mapsto G_{-[\lambda]} \bullet E(\lambda) \bullet G^{\bullet-1} \tag{4.1}$$

with an invertible $G \in \mathcal{A}$. If one finds such a transformation with a nontrivial dependence on ζ , it gives rise to a Bäcklund transformation. Corresponding examples and further explorations of (3.1) and its multi-component generalizations will be presented in a separate work.

¹⁴ Equation (3.1) can be interpreted as ‘discrete’ in the sense of [51]. See also [54, 55] for a general approach towards integrable systems via discrete zero curvature equations.

Appendix A. Some elementary notes on deformations of associative algebras

Although in the context of this work the question of *inequivalent* deformations of associative products addressed by the cohomological methods of deformation theory [56, 57] appears to be of minor importance, corresponding considerations help to find particular deformations (see also [15, 16] and [8] for the Lie algebra case). The following is an elementary approach and we refer the reader to the literature for more substantial treatments. Let \mathcal{A} be an associative algebra. Here, we are only looking for associative deformations depending *linearly* on a parameter, say ζ ,

$$A \bullet B := AB + \zeta \Phi(A, B), \quad (\text{A.1})$$

where Φ is a bilinear mapping $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$. Several examples appeared in the main part of this work. Associativity imposes the following conditions on Φ ,

$$A\Phi(B, C) - \Phi(AB, C) + \Phi(A, BC) - \Phi(A, B)C = 0 \quad (\text{A.2})$$

and

$$\Phi(A, \Phi(B, C)) = \Phi(\Phi(A, B), C). \quad (\text{A.3})$$

The last relation means that Φ defines an associative product, the other relation tells us that Φ is a coboundary. This is a *compatibility condition* for the two products. If Φ is a cocycle, so that there is a linear mapping $N : \mathcal{A} \rightarrow \mathcal{A}$ such that

$$\Phi(A, B) = N(A)B + AN(B) - N(AB), \quad (\text{A.4})$$

then (A.2) is solved and (A.3) takes the form

$$AT(B, C) - T(AB, C) + T(A, BC) - T(A, B)C = 0, \quad (\text{A.5})$$

where

$$T(A, B) := N(N(A)B + AN(B) - N(AB)) - N(A)N(B) \quad (\text{A.6})$$

is the Nijenhuis torsion of N . Thus, T has to be a coboundary. (A.5) in turn is solved if T is a cocycle, i.e., if there is a linear mapping $f : \mathcal{A} \rightarrow \mathcal{A}$ such that

$$T(A, B) = Af(B) - f(AB) + f(A)B. \quad (\text{A.7})$$

A simple solution is given by $f = 0$, which implies the *associative Nijenhuis relation* (see also [15, 58, 59], for example)

$$N(A)N(B) = N(AN(B) + N(A)B - N(AB)). \quad (\text{A.8})$$

Example 1. Let us fix an element $P \in \mathcal{A}$. Then,

$$N(A) = PA \quad (\text{A.9})$$

solves (A.8) and we have

$$\Phi(A, B) = APB. \quad (\text{A.10})$$

Example 2. Let $A \mapsto \hat{A}$ be an involution of \mathcal{A} , so that $\widehat{AB} = \hat{A}\hat{B}$. Setting

$$N(A) := \frac{1}{2}A^{(-)} \quad \text{where} \quad A^{(\pm)} := \frac{1}{2}(A \pm \hat{A}), \quad (\text{A.11})$$

T is a cocycle with $f(A) = -(1/8)A^{(-)}$ and

$$\Phi(A, B) = A^{(-)}B^{(-)}. \quad (\text{A.12})$$

Note that the involution leads to a direct sum decomposition $\mathcal{A} = \mathcal{A}^{(+)} \oplus \mathcal{A}^{(-)}$ where $\mathcal{A}^{(+)}$ is a subalgebra and $\mathcal{A}^{(+)}\mathcal{A}^{(-)} \subset \mathcal{A}^{(-)}$, $\mathcal{A}^{(-)}\mathcal{A}^{(+)} \subset \mathcal{A}^{(-)}$ and $\mathcal{A}^{(-)}\mathcal{A}^{(-)} \subset \mathcal{A}^{(+)}$. In particular, every \mathbb{Z}_2 -graded algebra has an obvious involution (see also [60] for the Lie algebra case). This includes the important cases of Clifford and super-algebras.

Appendix B. ‘Undeforming’ the product

Let us recall the definition of the product used in section 2.1.1 (see also example 1 in appendix A),

$$A \bullet B = APB + \zeta A(I - P)B \quad (\text{B.1})$$

where I is the unit matrix and $P^2 = P$. By a direct calculation one verifies that this product satisfies

$$Q(A \bullet B)Q = (QAQ)(QBQ) \quad (\text{B.2})$$

with

$$Q := P + z(I - P), \quad \zeta = z^2. \quad (\text{B.3})$$

For example, for 2×2 matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (\text{B.4})$$

we have

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}, \quad QAQ = \begin{pmatrix} a & zb \\ zc & z^2d \end{pmatrix}. \quad (\text{B.5})$$

With V given by (2.2) we associate

$$\hat{V} := QVQ = v_0 + v_1z^{-2} + v_2z^{-4} + \dots \quad (\text{B.6})$$

where the coefficients $v_n = Qu_nQ$ now exhibit an explicit z -dependence. The hierarchy equations

$$V_n = (\zeta^n V)_{\geq 0} \bullet V - V \bullet (\zeta^n V)_{\geq 0} \quad (\text{B.7})$$

can then be written in terms of \hat{V} as

$$\hat{V}_{t_n} = (z^{2n} \hat{V})_{\geq 0} \hat{V} - \hat{V} (z^{2n} \hat{V})_{\geq 0}, \quad (\text{B.8})$$

if we agree upon the rule that the projection $(\)_{\geq 0}$ does *not* take the ‘inner’ z -dependence of the v_n into account (i.e., the z -dependence shown in the last matrix in (B.5)). In this way, contact is made with previous formulations of hierarchies of the type considered in this work (see, in particular, [23, 24]). By use of deformed products one achieves a much more elegant formulation, according to our opinion (and remains in the class of ‘Gelfand–Dickey-type’ hierarchies).

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